Find $\frac{\partial^3 u}{\partial x^2} + \frac{\partial u}{\partial y^2}$ in Polar Coords. So assume $u(x, y)$.

where $x = r \cos \theta$

$y = r \sin \theta$

$x^2 + y^2 = r^2$

Now $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta} \frac{\partial \theta}{\partial x}$

$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial r^2} \left( \frac{\partial r}{\partial x} \right)^2 + \frac{\partial u}{\partial r} \frac{\partial^2 r}{\partial x^2} + \frac{\partial u}{\partial \theta} \frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 u}{\partial \theta^2} \frac{\partial \theta}{\partial x} \frac{\partial \theta}{\partial x}$

Similarly

$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} \left( \frac{\partial r}{\partial y} \right)^2 + \frac{\partial u}{\partial r} \frac{\partial^2 r}{\partial y^2} + \frac{\partial u}{\partial \theta} \frac{\partial^2 \theta}{\partial y^2} + \frac{\partial^2 u}{\partial \theta^2} \frac{\partial \theta}{\partial y} \frac{\partial \theta}{\partial y}$

$\frac{\partial^3 u}{\partial x^2} + \frac{\partial^3 u}{\partial y^2} = \frac{\partial^3 u}{\partial r^3} \left( \frac{\partial r}{\partial x} \right)^2 + \frac{\partial^2 u}{\partial r^2} \left( \frac{\partial^2 r}{\partial x^2} \right) + \frac{\partial u}{\partial r} \frac{\partial^3 r}{\partial x^3} + \frac{\partial \theta}{\partial x} \frac{\partial^2 \theta}{\partial x^2} + \frac{\partial u}{\partial \theta} \frac{\partial^3 \theta}{\partial x^3} + \frac{\partial^2 u}{\partial \theta^2} \frac{\partial \theta}{\partial x} \frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^3 u}{\partial \theta^3} \frac{\partial \theta}{\partial x} \frac{\partial^2 \theta}{\partial x^2}$

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we find that

$M_{xx} + M_{yy} = M_{rr} + \frac{\mu \phi}{r} + \frac{M}{r}$
Find \( \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial y^2} \) in Polar Coords. So assume \( u(r, \theta) \).

where \( x = r \cos \theta \)

\[ y = r \sin \theta \]

\[ x^2 + y^2 = r \]

Now \( \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta} \frac{\partial \theta}{\partial x} \]

\[ \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x} \left[ \frac{\partial u}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta} \frac{\partial \theta}{\partial x} \right] \]

\[ = \frac{\partial^2 u}{\partial r^2} \left( \frac{\partial r}{\partial x} \right)^2 + \frac{\partial^2 u}{\partial r \partial \theta} \frac{\partial \theta}{\partial x} \frac{\partial \theta}{\partial x} + \frac{\partial u}{\partial \theta} \frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 u}{\partial \theta^2} \left( \frac{\partial \theta}{\partial x} \right)^2 \]

Similarly

\[ \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} \left( \frac{\partial r}{\partial y} \right)^2 + \frac{\partial^2 u}{\partial r \partial \theta} \frac{\partial \theta}{\partial y} \frac{\partial \theta}{\partial y} + \frac{\partial u}{\partial \theta} \frac{\partial^2 \theta}{\partial y^2} + \frac{\partial^2 u}{\partial \theta^2} \left( \frac{\partial \theta}{\partial y} \right)^2 \]

\[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} \left( \frac{\partial r}{\partial x} \right)^2 + \frac{\partial^2 u}{\partial r \partial \theta} \frac{\partial \theta}{\partial x} \frac{\partial \theta}{\partial x} + \frac{\partial u}{\partial \theta} \frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 u}{\partial \theta^2} \left( \frac{\partial \theta}{\partial x} \right)^2 \]

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we find that

\[ M_{xx} + M_{yy} = M_{rr} + \frac{M_{r\theta}}{r} + \frac{M_{\theta\theta}}{r^2} \]