

```

[ > restart:
[ > with (student):with(plots):
  with(linalg):
Warning, the name changecoords has been redefined
Warning, the protected names norm and trace have been redefined and unprotected

```

```

ex 1
let's try y=mx

```

```

[ > f:= (x,y)-> (x^2*y^2)/(x^2+y^2);limit(f(x,y), {x=0,y=0});

```

$$f := (x, y) \rightarrow \frac{x^2 y^2}{x^2 + y^2}$$

$$\lim \left( \frac{x^2 y^2}{x^2 + y^2}, \{x = 0, y = 0\} \right)$$

```

[ > y:= m*x: limit(((m*x)^2*x^2)/(x^2+m^2*x^2), {x=0,y=0});
Error, (in limit) invalid limiting point

```

```

[ > f:= (x,y)-> (2 *x*y)/(x^2+y^2);limit(f(x,y), {x=0,y=0});
[ > y:= m*x: limit((m*x*2*x)/(x^2+m^2*x^2), {x=0});

```

$$f := (x, y) \rightarrow 2 \frac{x y}{x^2 + y^2}$$

```

Error, (in limit) invalid limiting point

```

$$2 \frac{m}{1 + m^2}$$

```

ex2

```

```

[ > restart:
[ > with (student):with(plots):
  with(linalg):
[ > f:=(x,y)-> x^y ;
Warning, the name changecoords has been redefined
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$$f := (x, y) \rightarrow x^y$$

```

[ > x/y*Diff(f(x,y),x) +1/(ln(x))*Diff(f(x,y),y)=x/y*difff(f(x,y),x)
  +1/(ln(x))*diffe(f(x,y),y);

```

$$\frac{x \left( \frac{\partial}{\partial x} x^y \right)}{y} + \frac{\frac{\partial}{\partial y} x^y}{\ln(x)} = 2 x^y$$

```

ex3

```

```
[ > restart:
> with (student):with(linalg):
Warning, the protected names norm and trace have been redefined and unprotected
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```
[ > v:=array(1..2,[2,1]);F:=(x,y)-> x^2*y+sqrt(y);
```

$$v := [2, 1]$$

$$F := (x, y) \rightarrow x^2 y + \sqrt{y}$$

```
[ > Diff(F(x,y),x)=diff(F(x,y),x);Diff(F(x,y),y)=diff(F(x,y),y);
```

$$\frac{\partial}{\partial x}(x^2 y + \sqrt{y}) = 2 x y$$

$$\frac{\partial}{\partial y}(x^2 y + \sqrt{y}) = x^2 + \frac{1}{2\sqrt{y}}$$

```
[ > fx:=subs(x=2,y=1,diff(F(x,y),x));
```

$$fx := 4$$

```
[ > fy:=subs(x=2,y=1,diff(F(x,y),y));
```

$$fy := \frac{9}{2}$$

```
[ > grad:=array(1..2,[fx,fy]);norm(grad,2);
```

$$grad := \left[ 4, \frac{9}{2} \right]$$

$$\frac{1}{2}\sqrt{145}$$

```
[ ex #4
```

```
[ > restart:with (student):
> f:=(x,y)-> 3*x*y-x^2*y-x*y^2;
```

$$f := (x, y) \rightarrow 3 y x - x^2 y - x y^2$$

```
[ > Diff(f(x,y),x)=diff(f(x,y),x);Diff(f(x,y),y)=diff(f(x,y),y);
```

$$\frac{\partial}{\partial x}(3 y x - x^2 y - x y^2) = 3 y - 2 y x - y^2$$

$$\frac{\partial}{\partial y}(3 y x - x^2 y - x y^2) = 3 x - x^2 - 2 y x$$

```
[ > fx:=(x,y)->diff(f(x,y),x);fy:=(x,y)->diff(f(x,y),y);
```

$$fx := (x, y) \rightarrow \text{diff}(f(x, y), x)$$

$$fy := (x, y) \rightarrow \text{diff}(f(x, y), y)$$

```
[ > fxy:=diff(fx(x,y),y);fxx:=diff(fx(x,y),x);fyy:=diff(fy(x,y),y);
```

$$fxy := 3 - 2 x - 2 y$$

$$fxx := -2 y$$

$$fyy := -2 x$$

```

> xf:=diff(f(x,y),x);yf:=diff(f(x,y),y); solve({xf=0,yf=0},{x,y});
      xf:=3y-2yx-y^2
      yf:=3x-x^2-2yx
      {y=0,x=0},{y=0,x=3},{x=0,y=3},{y=1,x=1}
> disc:=(x,y)->
diff(f(x,y),x,x)*diff(f(x,y),y,y)-diff(f(x,y),x,y)*diff(f(x,y),x,y)
);DXY:=fxx*fyy-fxy*fxy;
      disc:=(x,y) -> diff(f(x,y),x,x) diff(f(x,y),y,y) - diff(f(x,y),x,y)^2
      DXY:=4yx-(3-2x-2y)^2
> subs(x=0,y=0,disc);
      disc
> subs(x=3,y=0,disc);
      disc
> subs(x=0,y=3,disc);
      disc
> subs(x=1,y=1,disc);
> subs(x=1,y=1,fxx);#local max
      disc
      -2

```

ex 5

```

> restart:with(student):z:=(x,y)->cos(x*y)+y*cos(x);
      z:=(x,y) -> cos(yx) + y cos(x)
> X:=(u,v)->u^2+v;Y:=(u,v)->u-v^2;
      X:=(u,v) -> u^2 + v
      Y:=(u,v) -> u - v^2
> Diff(z(x,y),x)*Diff(X(u,v),u)+Diff(z(x,y),y)*Diff(Y(u,v),u)=
diff(z(x,y),x)*diff(X(u,v),u)+diff(z(x,y),y)*diff(Y(u,v),u);

$$\left(\frac{\partial}{\partial x}(\cos(yx) + y \cos(x))\right)\left(\frac{\partial}{\partial u}(u^2 + v)\right) + \left(\frac{\partial}{\partial y}(\cos(yx) + y \cos(x))\right)\left(\frac{\partial}{\partial u}(u - v^2)\right) =$$


$$2(-\sin(yx)y - y \sin(x))u - \sin(yx)x + \cos(x)$$

> subs(x=u^2+v,y=u-v^2,%);simplify(%);
Diff(cos((u-v^2)(u^2+v))+(u-v^2)cos(u^2+v),u^2+v)\left(\frac{\partial}{\partial u}(u^2+v)\right)
+ Diff(cos((u-v^2)(u^2+v))+(u-v^2)cos(u^2+v),u-v^2)\left(\frac{\partial}{\partial u}(u-v^2)\right)=

$$2(-\sin((u-v^2)(u^2+v))(u-v^2) - (u-v^2)\sin(u^2+v))u - \sin((u-v^2)(u^2+v))(u^2+v) + \cos(u^2+v)$$


```

$$\begin{aligned} & \text{Diff}(\cos((-u+v^2)(u^2+v)) + \cos(u^2+v)u - \cos(u^2+v)v^2, u^2+v) \left( \frac{\partial}{\partial u}(u^2+v) \right) \\ & + \text{Diff}(\cos((-u+v^2)(u^2+v)) + \cos(u^2+v)u - \cos(u^2+v)v^2, u-v^2) \left( \frac{\partial}{\partial u}(u-v^2) \right) = \\ & 3 \sin((-u+v^2)(u^2+v))u^2 - 2u \sin((-u+v^2)(u^2+v))v^2 - 2 \sin(u^2+v)u^2 \\ & + 2u \sin(u^2+v)v^2 + \sin((-u+v^2)(u^2+v))v + \cos(u^2+v) \end{aligned}$$

>

> **Diff(z(x,y),x)\*Diff(X(u,v),v)+Diff(z(x,y),y)\*Diff(Y(u,v),v)=  
diff(z(x,y),x)\*diff(X(u,v),v)+diff(z(x,y),y)\*diff(Y(u,v),v);**

$$\begin{aligned} & \left( \frac{\partial}{\partial x}(\cos(yx) + y \cos(x)) \right) \left( \frac{\partial}{\partial v}(u^2+v) \right) + \left( \frac{\partial}{\partial y}(\cos(yx) + y \cos(x)) \right) \left( \frac{\partial}{\partial v}(u-v^2) \right) = \\ & -\sin(yx)y - y \sin(x) - 2(-\sin(yx)x + \cos(x))v \end{aligned}$$

> **subs(x=u^2+v,y=u-v^2,%);simplify(%);**

$$\begin{aligned} & \text{Diff}(\cos((u-v^2)(u^2+v)) + (u-v^2)\cos(u^2+v), u^2+v) \left( \frac{\partial}{\partial v}(u^2+v) \right) \\ & + \text{Diff}(\cos((u-v^2)(u^2+v)) + (u-v^2)\cos(u^2+v), u-v^2) \left( \frac{\partial}{\partial v}(u-v^2) \right) = \\ & -\sin((u-v^2)(u^2+v))(u-v^2) - (u-v^2)\sin(u^2+v) \\ & - 2(-\sin((u-v^2)(u^2+v))(u^2+v) + \cos(u^2+v))v \end{aligned}$$

$$\begin{aligned} & \text{Diff}(\cos((-u+v^2)(u^2+v)) + \cos(u^2+v)u - \cos(u^2+v)v^2, u^2+v) \left( \frac{\partial}{\partial v}(u^2+v) \right) \\ & + \text{Diff}(\cos((-u+v^2)(u^2+v)) + \cos(u^2+v)u - \cos(u^2+v)v^2, u-v^2) \left( \frac{\partial}{\partial v}(u-v^2) \right) = \\ & \sin((-u+v^2)(u^2+v))u - 3 \sin((-u+v^2)(u^2+v))v^2 - \sin(u^2+v)u + \sin(u^2+v)v^2 \\ & - 2v \sin((-u+v^2)(u^2+v))u^2 - 2v \cos(u^2+v) \end{aligned}$$

>

ex6

> **restart:with(student):with(linalg):**

Warning, the protected names norm and trace have been redefined and unprotected

> **F:=(x,y)->x\*exp(x)+cos(x\*y);**

$$F := (x, y) \rightarrow x e^x + \cos(yx)$$

$$v := [2, 1]$$

$$F := (x, y) \rightarrow x^2 y + \sqrt{y}$$

> **grad(F(x,y), vector([x,y]));subs(x=2,y=0,%);dotprod(%,[3,-4]/5);**

$$[e^x + x e^x - \sin(yx)y, -\sin(yx)x]$$

$$[3 e^2, -2 \sin(0)]$$

$$\frac{9}{5}e^2 + \frac{8}{5}\sin(0)$$

[ ex7

> **restart:with (student) :with(linalg) :with (plots) :**

Warning, the protected names norm and trace have been redefined and unprotected

Warning, the name changecoords has been redefined

> **F:=(x,y,z) ->**

**x^2+y^2+z-9;plot1:=implicitplot3d(F(x,y,z),x=-3..3,y=-3..3,z=-9..9)**  
**) :**

$$F := (x, y, z) \rightarrow x^2 + y^2 + z - 9$$

> **grad(F(x,y,z),**

**vector([x,y,z]));subs(x=1,y=2,z=4,%);dotprod(%,[x-1,y-2,z-4])=0;**

>

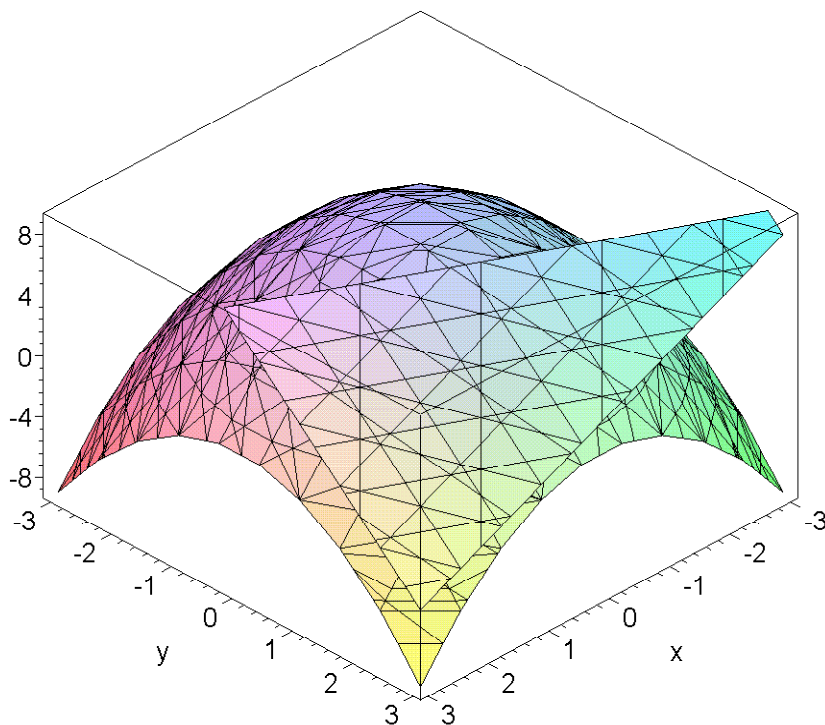
> **plot2:=implicitplot3d(-14+2\*x+4\*y+z,x=-3..3,y=-3..3,z=-9..9) :**

> **display(plot1,plot2,axes=boxed) ;**

$$[2x, 2y, 1]$$

$$[2, 4, 1]$$

$$-14 + 2x + 4y + z = 0$$



[ ex #8

```

> restart:with(student):
> f:=(x,y)-> -x^2+x*y-2*y-2*x-y^2+4;
      f:=(x,y) → -x2 + yx - 2y - 2x - y2 + 4
> Diff(f(x,y),x)=diff(f(x,y),x);Diff(f(x,y),y)=diff(f(x,y),y);
      ∂
      ∂x (-x2 + yx - 2y - 2x - y2 + 4) = -2x + y - 2
      ∂
      ∂y (-x2 + yx - 2y - 2x - y2 + 4) = x - 2 - 2y
>
      fxx := -2y
      fyy := -2x
> xf:=diff(f(x,y),x);yf:=diff(f(x,y),y); solve({xf=0,yf=0},{x,y});
      xf := -2x + y - 2
      yf := x - 2 - 2y
      {y = -2, x = -2}
> diff(f(x,y),x,x)*diff(f(x,y),y,y) -
diff(f(x,y),x,y)*diff(f(x,y),x,y);
      3

```

```

>
ex 9

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```

> restart:with(student):w:=(x,y,z)->(x*y)+z;
      w:=(x,y,z) → yx + z
> X:=x->cos(t);Y:=y->sin(t);Z:=z->t;
      X:=x → cos(t)
      Y:=y → sin(t)
      Z:=z → t
> Diff(w(x,y,z),x)*Diff(X(t),t)+Diff(w(x,y,z),y)*Diff(Y(t),t)
+Diff(w(x,y,z),z)*Diff(Z(t),t)=
diff(w(x,y,z),x)*diff(X(t),t)+diff(w(x,y,z),y)*diff(Y(t),t)
+diff(w(x,y,z),z)*diff(Z(t),t);
      (∂
      ∂x (yx + z)) (∂
      ∂t cos(t)) + (∂
      ∂y (yx + z)) (∂
      ∂t sin(t)) + (∂
      ∂z (yx + z)) (∂
      ∂t t) =
      -y sin(t) + x cos(t) + 1
> subs(x=cos(t),y=sin(t),z=t,%);simplify(%);
Diff(sin(t) cos(t) + t, cos(t)) (∂
∂t cos(t)) + Diff(sin(t) cos(t) + t, sin(t)) (∂
∂t sin(t))
+ (∂
∂t (sin(t) cos(t) + t)) (∂
∂t t) = -sin(t)2 + cos(t)2 + 1

```

$$\begin{aligned} & \text{Diff}(\sin(t) \cos(t) + t, \cos(t)) \left( \frac{\partial}{\partial t} \cos(t) \right) + \text{Diff}(\sin(t) \cos(t) + t, \sin(t)) \left( \frac{\partial}{\partial t} \sin(t) \right) \\ & + \left( \frac{\partial}{\partial t} (\sin(t) \cos(t) + t) \right) \left( \frac{\partial}{\partial t} t \right) = 2 \cos(t)^2 \end{aligned}$$

[ >

[ >

[

[ >

ex13

> **restart:with (student):with(linalg):with(plots):**

>

Warning, the protected names norm and trace have been redefined and unprotected

Warning, the name changecoords has been redefined

> **f:=(x,y,z) -> x^2/9+y^2+z^2/9-3;**

>

> **grad(f(x,y,z),**  
**vector([x,y,z])); subs(x=-3,y=1,z=-3,%); dotprod(%,[x+3,y-1,z+3])=0;**

$$f := (x, y, z) \rightarrow \frac{1}{9}x^2 + y^2 + \frac{1}{9}z^2 - 3$$

$$\left[ \frac{2}{9}x, 2y, \frac{2}{9}z \right]$$

$$\left[ \frac{-2}{3}, 2, \frac{-2}{3} \right]$$

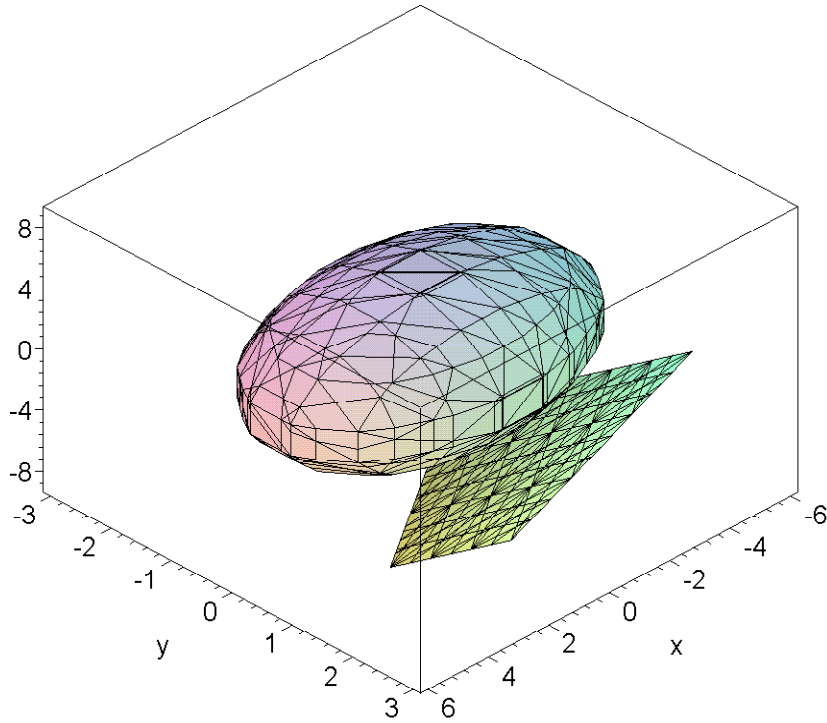
$$-6 - \frac{2}{3}x + 2y - \frac{2}{3}z = 0$$

[ > **plot1:=implicitplot3d(f(x,y,z),x=-6..6,y=-3..3,z=-9..9):**

[

[ > **plot2:=implicitplot3d(-6-2\*x/3+2\*y-2\*z/3,x=-3..3,y=-3..3,z=-9..9):**

> **display(plot1,plot2,axes=boxed);**



ex 14

```
> restart:with (student):U:= (x,y,z)->x^4*y+y^2*z^3;
```

$$U := (x, y, z) \rightarrow x^4 y + y^2 z^3$$

```
> X:= (r,s,t)-> r*s*exp(t);Y:= (r,s,t)-> r*s^2*exp(-t);Z:= (r,s,t)->
r*s*sin(t);
```

$$X := (r, s, t) \rightarrow r s e^t$$

$$Y := (r, s, t) \rightarrow r s^2 e^{-t}$$

$$Z := (r, s, t) \rightarrow r s \sin(t)$$

```
> Diff(U(x,y,z),x)*Diff(X(r,s,t),s)+Diff(U(x,y,z),y)*Diff(Y(r,s,t),s)
)+Diff(U(x,y,z),z)*Diff(Z(r,s,t),t)=diff(U(x,y,z),x)*diff(X(r,s,t),
s)+diff(U(x,y,z),y)*diff(Y(r,s,t),s)+diff(U(x,y,z),z)*diff(Z(r,s,
t),t);diff(U(x,y,z),x)*diff(X(r,s,t),s)+diff(U(x,y,z),y)*diff(Y(r,
s,t),s)+diff(U(x,y,z),z)*diff(Z(r,s,t),t);
```

$$\left(\frac{\partial}{\partial x}(x^4 y + y^2 z^3)\right)\left(\frac{\partial}{\partial s} r s e^t\right) + \left(\frac{\partial}{\partial y}(x^4 y + y^2 z^3)\right)\left(\frac{\partial}{\partial s} r s^2 e^{-t}\right) \\ + \left(\frac{\partial}{\partial z}(x^4 y + y^2 z^3)\right)\left(\frac{\partial}{\partial t} r s \sin(t)\right) = 4 x^3 y r e^t + 2 (x^4 + 2 y z^3) r s e^{-t} + 3 y^2 z^2 r s \cos(t)$$

$$4 x^3 y r e^t + 2 (x^4 + 2 y z^3) r s e^{-t} + 3 y^2 z^2 r s \cos(t)$$

```
> subs(x=r*s*exp(t),y=r*s^2*exp(-t),z=r*s*sin(t),%);
```



```

[      4 r^5 s^5 (e^t)^4 e^(-t) + 2 (r^4 s^4 (e^t)^4 + 2 r^4 s^5 e^(-t) sin(t)^3) r s e^(-t) + 3 r^5 s^7 (e^(-t))^2 sin(t)^2 cos(t)
[ > simplify(%);
[      -r^5 s^5 e^(-t) (-6 e^(4t) - 4 s e^(-t) sin(t) + 4 s e^(-t) sin(t) cos(t)^2 - 3 s^2 e^(-t) cos(t) + 3 s^2 e^(-t) cos(t)^3)
[ ex 15

```

```

[ > restart:with (student):with(linalg):T:= (x,y,z)->80/(1 + x^2
[      +2*y^2+3*z^2);
[ Warning, the protected names norm and trace have been redefined and unprotected

```

$$T := (x, y, z) \rightarrow 80 \frac{1}{1 + x^2 + 2y^2 + 3z^2}$$

```

[ > v:=subs (x=1,y=1,z=1,grad(T(x,y,z), vector([x,y,z])));

```

$$v := \left[ \frac{-160}{49}, \frac{-320}{49}, \frac{-480}{49} \right]$$

```

[ > norm(v,2);

```

$$\frac{160}{49} \sqrt{14}$$

```

[ ex 16

```

```

[ > restart:with (student):with(linalg):with(plots):

```

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[ Warning, the protected names norm and trace have been redefined and unprotected
[ Warning, the name changecoords has been redefined

```

```

[ >
[ > f:=(x,y,z)->x^2 + 3*x*y-y^2 -z;

```

$$f := (x, y, z) \rightarrow x^2 + 3yx - y^2 - z$$

```

[ >
[ > plot1:=implicitplot3d(f(x,y,z),x=-3..3,y=-3..3,z=-9..9):v:=vector(
[      [x,y,z]);

```

$$v := [x, y, z]$$

```

[ > grad(f(x,y,z),v);

```

$$[2x + 3y, 3x - 2y, -1]$$

```

[ >
[ > subs (x=2,y=3,z=18,%);

```

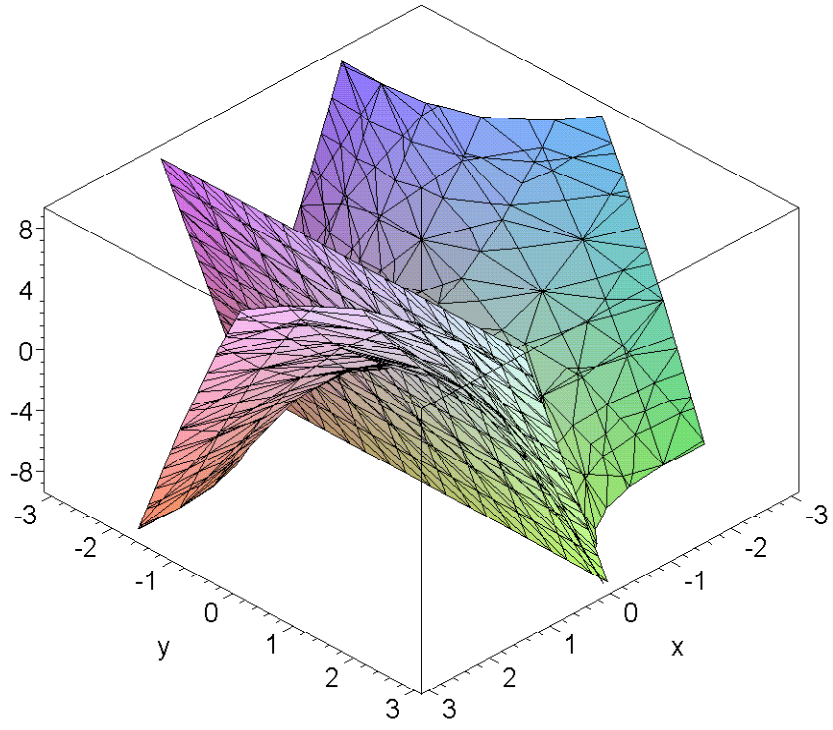
$$[13, 0, -1]$$

```

[ >

```

```
[ > dotprod(%, [x-2, y-3, z-18])=0;  
      -8 + 13x̄ - z̄ = 0  
>  
[ > plot2:=implicitplot3d(-8+13*x-z, x=-3..3, y=-3..3, z=-9..9) :  
[ > display(plot1, plot2, axes=boxed) ;
```



```
[ >
```