1a.) Two distinct vectors \( x_1 \) and \( x_2 \) are solutions to \( Ax = b \). Prove that \( N(A) \) is non-zero.

1b.) Let \( B \) be a basis for \( \mathbb{R}^n \). Prove the following:
   i.) \( B \) does not contain the zero vector.
   ii.) Any proper subset of \( B \) is NOT a basis.
   iii.) No vector in \( B \) is a linear combination of the other vectors in \( B \).

1c.) If \( A \) is a 3x3 matrix such that

\[
\begin{pmatrix}
0 \\
1 \\
2
\end{pmatrix} = \begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix} \quad \text{and} \quad
\begin{pmatrix}
3 \\
4 \\
5
\end{pmatrix} = \begin{pmatrix}
0 \\
1 \\
0
\end{pmatrix}
\]

then find \( A \begin{pmatrix}
6 \\
7 \\
8
\end{pmatrix} \).

2a.) The solution to

\[
ax + ay - z = 1 \\
x - ay - az = -1 \\
ax - y + az = 1
\]

is \((x, y, z) = (a, b, a)\) Find \( a + b \).

2b.) Describe the values for \( k \) for which the following system have no, unique or infinite number of solutions.

\[
kx + y + z = 1 \\
x + ky + z = k \\
x + y + kz = k^2.
\]

2c.) Consider the system

\[
x + y + z = 0 \\
x + 2y + 3z = 0 \\
x + 3y + bz = 0
\]

where \( b \) is a constant. Find the value of \( b \) such that:
   i.) the system has a unique solution.
   ii.) the system has more than one solution.
   iii.) the system has NO solution.

3.) Let \( A, B, C \) be 2x2 matrices and \( O \) the zero matrix:

Prove or disprove:
   i.) If \( A^2 = O \) then \( A = O \).
   ii.) If \( AB = AC \) then \( B = C \).
   iii.) If \( A \) is invertible and \( A = A^{-1} \) then either \( A = I \) or \( A = -I \).

4.) Find \( a, b, c \) such that

\[
\begin{pmatrix}
3 & -2 & -2 \\
-1 & 1 & 1 \\
3 & -1 & 1
\end{pmatrix}
\]

is the inverse of

\[
\begin{pmatrix}
1 & a & 0 \\
-1 & b & 1 \\
2 & c & -1
\end{pmatrix}.
\]
5.) Find $k$ such that the vectors $v_1, v_2, v_3$ form a basis for $R^3$ where $v_1 = (-1,1,1), v_2 = (1,1,1)$ and $v_3 = (1,-1,k)$.

6.) Find the row rank, column rank and determinant for the following matrix:

$$\begin{pmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9
\end{pmatrix}.$$ 

ii. Define the function $f : R^3 \rightarrow R^3$ by $f(x) = A \times x$. What is the range of $f$ in $R^3$.

7.) Find the values of $k$ for which the following matrix is invertible:

$$\begin{pmatrix}
7 & 6 & 0 \\
5 & 4 & x \\
8 & 7 & 0
\end{pmatrix}.$$ 

8.) Find $k$ for which $b = (12,11,k)$ is in the range of

$$\begin{pmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9
\end{pmatrix}.$$ 

9. Find the dimension in $R^4$ of the subspace spanned by

$\{ (1,-1,0,1), (-2,1,1,1), (-1,0,1,2), (1,1,-2,-5) \}$.

10.) Describe the collection of points in $R^2$ which satisfy $Ax = 0$ where $A$ is the following:

$$\begin{pmatrix}
x & y & 1 \\
0 & y & l \\
0 & y & 1
\end{pmatrix}.$$ 

11.) Prove the collection of real symmetric matrices are a subspace of $M_{3x3}$.

12.) Let $T : R^2 \rightarrow R^2$ be a linear transformation such that $T(1,2) = (-1,1)$ and $T(0,-1) = (2,-1)$.
   i.) Find a matrix which represents $T$.
   ii.) Find $T(1,1)$.

13.) Let $T : R^2 \rightarrow R^2$ be a linear transformation such that $T(1,2) = (2,3)$ and $T(-1,2) = (2,-3)$.
   i.) Find a matrix which represents $T$.
   ii.) Find $T(2,1)$.

14.) Let $T,S : R^2 \rightarrow R^2$ be the respectively operators that rotate each vector counterclockwise $90^\circ$ and reflects each vector through the $y-axis$.
   i.) Find $T \circ S$. 

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ii.) Find $S \circ T$.

iii.) Find the matrices which represents $T$ and $S$ with respect to the standard basis.

15.) Find the eigenvalues of the matrix

$$
\begin{pmatrix}
2 & b \\
3 & -1
\end{pmatrix}.
$$

i. If the eigen values are $-4$ and $b - 1$, find $b$.

16.) Find the eigenvalues and normalized eigenvectors $p_1, p_2$ of the matrix

$$
\begin{pmatrix}
2 & 1 \\
1 & 2
\end{pmatrix}.
$$

i. orthogonally diagonalize $A$.

17.) If $A$ is an $nxn$ invertible matrix. Prove the if $\lambda$ is an eigenvalue of $A$ then $\lambda^{-1}$ is an eigenvalue of $A^{-1}$.

18.) If $A$ is an $nxn$ matrix with eigenvector $v$.

i.) $v$ is an eigenvector of $2A$.

ii.) $v$ is an eigenvector of $A^2$.

iii.) $v$ is an eigenvector of $A^{-1}$.

19.) Let $w$ be an $n - vector$ and $A = w \ast w^\top$.

i.) Prove that $A$ is symmetric.

ii.) Prove that $A$ has an eigenvalue equal to $w^\top \ast w$ with eigenvector $w$.

iii.) Prove that the remaining $n - 1$ eigenvalues are 0 and span $w^\perp$. 