Math 3751 Material for Final Exam

THE EXAM There are eight questions.
(1) Definitions. (15 pts) You will be asked to give the exact definitions for four items. Absolute accuracy is required.
(2) Statements of theorems. (15 pts) You will be asked to give the exact statements (as given in your textbook) of three theorems. Absolute accuracy is required.
(3), (4), and (5) Proofs of Theorems. (10 pts each, for a total of 30 pts) You will be asked to give the proof for three theorems taken from this list: (6), (7), and (8) Problem solving. (40 pts) These are questions that can be done by using what has been discussed in the course.

I. Define the following terms or state the following theorems.

The Lagrange Mean value Theorem and the Cauchy Mean Value Theorem.
\( f(x) \) is integrable on \([a, b]\).
State carefully the two fundamental theorems of Calculus.
Darboux’s Theorem (p.94).
The Lagrange Remainder Theorem (p.169).
Convergence of a Taylor Series (p. 175).

II. Prove or maybe disprove

1.)
   a.) If \( f : [a, b] \rightarrow R \) has infinitely many discontinuities but is integrable on \([a, b]\).

   b.) A non-integrable function.

2.) Working directly from the definition, prove that a piecewise constant function is integrable.

3.) Give an example of an integrable function \( f(x) \) such that \( f(x) \geq 0 \) and \( \int_{a}^{b} f = 0 \) but \( f \neq 0 \) What happens when \( f(x) \) is continuous. Prove it.
Prove directly from the definition that a bounded monotone function is integrable.

4.) Use the Fundamental Theorem of Calculus to calculate $\int_{1}^{3}(3x - 1) \, dx$. Give reasons why the hypothesis of the Fundamental Theorem of calculus are satisfied.

Let $n$ be a natural number. Find a function $f$ and a partition $P_n$ such that $\sum_{i=1}^{n} \frac{i^2}{n^3}$ is a Riemann sum based on $P_n$. What is the sum?

5.) Let $f(x) = \begin{cases} x - 3 & \text{if } x \geq 1 \\ x^2 - 2x - 1 & \text{if } x < 1 \end{cases}$

Prove directly that $f$ is integralable by finding a sequence of partitions $P_n$ of $[1, 3]$ such that $U(f, P_n) - L(f, P_n) < \frac{1}{n}$.

6.) For the function $f(x) = \begin{cases} x - 3 & \text{if } x \geq 1 \\ x^2 - 2x - 1 & \text{if } x < 1 \end{cases}$

and the partition $P = \{ 1, 3/2, 3/4 \}$ find the Darboux Sums $U(f, P)$ and $L(f, P)$.

7.) For $f(t)$ as in 6 find:

$F(x) = \int_{0}^{x} f(t) dt$ where $0 \leq x \leq 2$ This is a computation like Homework 9.

$\frac{d}{dx} \int_{0}^{\phi(x)} f(t) dt$.

8.) be able to prove theorem 6.9 and theorem 7.1 and theorem 8.11

III. Taylor Series Stuff

1.) Theorem 8.6 and Proposition 8.7: The irrationality of $e$ (p.170-171).

2.) Theorem 8.8.

3.) do some Taylor series or polynomials with remainder for well-known functions like $e^x$, $\cos(x)$ a typical polynomial.

4.) Remainder statement.