**Ex. 1**

Assume $W_1 \subseteq W_2 \subseteq W_3 \subseteq W_4 = \mathbb{R}^4$

- $W_1 = \ker(A-\lambda I) \quad \text{dim} W_1 = 5 = d_1$
- $W_2 = \ker((A-\lambda I)^2) \quad \text{dim} W_2 = 9 = d_2$
- $W_3 = \ker((A-\lambda I)^3) \quad \text{dim} W_3 = 12 = d_3$
- $W_4 = \ker((A-\lambda I)^4) = \{(A-\lambda I)^4 = 0\} \quad \text{dim} W_4 = 14$

Select a basis for $\mathbb{R}^4$ such that

$$\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}\}$$

Start with $v_1, v_2, v_4$ are in $\mathbb{R}^4$, but not in $W_3$. Call the $v(1,4), v(2,4)$

- Next level consider $v(1,3) = \text{Tor}(1,4)$
- $v(2,3) = \text{Tor}(2,4)$

Now $v(1,3), v(2,3) \in W_3 \setminus W_2$

Set as basis for $W_3$ as follows:

$$\{v_1, v_2, v_3, v(1,3), v(2,3)\}$$

Need an additional element, select $v(3,3)$

so that $v(3,3)$ added to $\{v_1, v_2, v_3, v(1,3), v(2,3), v(3,3)\}$ is a basis for $W_3$.

Now $v(1,4), v(2,4)$

- Power vectors $v(1,4), v(2,4), v(3,3)$
- $\text{Tor}(1,4), \text{Tor}(2,4), \text{Tor}(3,3)$
- $\text{Tor}(1,4), \text{Tor}(2,4), \text{Tor}(3,3)$

$\text{Tor}(1,4), \text{Tor}(2,4), \text{Tor}(3,3)$

Now consider $-v(3,3)$

$$\{v_1, v_2, v_3, -v(1,3), v(2,3), v(3,3)\}$$

in $W_3$ is a basis.

This is $\{v(1,3), v(2,3), v(3,3)\}$

Next consider $v(4,2)$

Need another to form a basis $v(4,2)$.

$v(4,2)$ is another cyclic vector.