Math 5820  Takehome Exam Test I  Due Friday 11/5/2004  Directions: You may
discuss the problems among yourselves and with me; however, the write up must be
your own.

I. Do the following:
1. What is the Jordan Form of an operator on a finite dimensional comles vector space? What
problems would occur if one restricted themselves to Real Spaces?

2.) Outline the procedure to find a Jordan Form of a matrix.

3.) Find all eigenvectors of
\[
A = \begin{pmatrix} 6 & -1 \\ 16 & -2 \end{pmatrix}
\]
What is its Jordan Form? Find the basis in \( \mathbb{R}^2 \) that
does the JOB!

4.)
 i.) Prove that If \( A \) has a square root e.g \( X^2 = A \) then any matrix similar to \( A \) has a square
root.
 ii.) Find the binomial series for \( (1 + t)^{1/2} \).
 iii.) Use this idea to compute the square root of a matrix whose Jordan Form is \( I + N \) where
\( N \) is nilpotent i.e. \( N^k = 0 \)

5.) Let \( a_n \) be the sequence of numbers defined by the sequence \( a_{n+2} = a_n + a_{n+1} \) where \( a_0 = 1, a_1 = 1 \)
 i.) Show that if \( T : (x, y) \rightarrow (y, x + y) \) then \( T : (a_{n-2}, a_{n-1}) \rightarrow (a_{n-1}, a_n) \) where \( a_n = a_{n-2} + a_{n-1} \) is the Fibonacci sequence,i.e. the sum of the previous 2 terms.
 ii.) Find \( T^n v \).
 iii.) Find the eigenvectors \( T \).
 iv.) Use this to find \( \lim_{n \to \infty} T^n \)

6.) Recall the definition of \( e^A \) where \( A \) is an \( n \times n \) matrix.
 i.) Prove that if the \( n \times n \) \( A \) and \( B \) commute then \( e^A e^B = e^{A+B} \).
 ii.) Find \( e^A \) where \( A = \begin{pmatrix} 6 & -1 \\ 16 & -2 \end{pmatrix} \).

7.) Let \( V = \mathbb{R}^3 \) and \( W = \mathbb{R}^2 \), let \( \{v_1, v_2, v_3\} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \). and let \( \{w_1, w_2\} \) be the
standard basis.
 a.) If \( T : V \rightarrow W \) is given by \( T(x, y, z) = (x + y, y + z) \). Find \( [T]_{v}^{w} \)
 b.) For \( \{v'_1, v'_2, v'_3\} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \). and let \( \{w_1, w_2\} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \) Find \( [T]_{v'}^{w'} \)
 c.) Find the change of basis matrix \( P \) from \( v_i \) to \( v'_i \) and \( Q \) from \( w_i \) to \( w'_i \)

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8.) If \( T : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is given by \( T(x_1, x_2, ..., x_n) = (x_n, x_{n-1}, ..., x_1) \). Find the characteristic and minimal polynomial.

9.) Let \( A \) be a 3x3 matrix and \( \{p_1, p_2, p_3\} \) be a Jordan chain associated with the eigenvalue \( c \) where \( (A - c \cdot I)p_1 = 0 \) and \( (A - c \cdot I)p_i = p_{i-1} \) for \( i = 2, 3 \).
   a. What is the Jordan form.
   b. Prove directly that \( A \cdot P = J \cdot P \) where \( J \) is the respective Jordan form and \( P = \{p_1, p_2, p_3\} \).