I. Basic Definitions, concepts

7.) prove triangle inequality.
3) determination of a subspace.
4) orthogonal vectors.
5) Determination of a span of vectors.

1.) Properties of Inverses.
4) connection between the inverse of a matrix and uniqueness of solution.
5) properties of dot real/complex product.
7) prove that the span is a subspace.

1. b. Is the set \( S = \{(1,0,1,0),(0,1,-1,2),(0,2,2,1),(1,0,0,1)\} \) a basis for \( \mathbb{R}^4 \)

2. a. To find a basis for \( \mathbb{R}^3 \) that includes the vectors \( \{(1,0,2),(0,1,3)\} \) one does the following: Explain.

b. Write each of \( e_1, e_2, e_3 \) in terms of this basis.

3.) Prove that the coordinates of a vector \( x \) with respect to a basis in \( \mathbb{R}^3 \) are unique.

4.) Find a basis for the homogenous system \( (\lambda I - A)x = 0 \) where \( \lambda = 1 \) and

\[
A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -3 \\ 0 & 1 & 3 \end{pmatrix}.
\]

Hint:

\[
\text{rref} \begin{pmatrix} 1 & -1 & 2 \\ 1 & 0 & -1 \\ 0 & -1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -3 \end{pmatrix}.
\]

5. Let \( S = \{x_1, x_2, x_3\} \) where \( x_1 = (1,0,2), x_2 = (-2,0,1), \) and \( x_3 = (0,\frac{1}{2},0) \).

a. Prove that \( S \) is an orthogonal basis.

b. Make \( S \) into an orthonormal basis.

c. Express \( x = (3,4,5) \) in terms of the orthonormal basis.

6.) Let \( S = \{(1,1),(2,3)\} \) and \( x = (1,5) \).

a. Prove that \( S \) is a basis.

b. Express the elements of \( S \) in terms of the natural basis \( \{e_1, e_2\} \).

c. Express the elements of the natural basis in terms of the \( S \) basis.

d. Relate c. to b.
e. compute the coefficients of \( x \) with respect to the \( S \) basis.

9.) construct an orthonormal basis from for the subspace spanned by \((1, 1, 1), (0, 0, 1), (1, 2, 3)\).

1.) Let \( S = \{x_1, x_2, x_3, x_4\} \) be a set of nonzero vectors in \( \mathbb{R}^4 \). If \( x_4 \) is a linear combination of \( x_1, x_2, x_3 \) then prove that \( S \) is linearly dependent.

6.) Let \( A \) be any \( n \times n \) matrix whose columns are linearly independent. Prove that \( Ax = 0 \) has a unique solution.

7.) Prove that the coordinates of a vector \( x \) with respect to a basis in \( \mathbb{R}^4 \) are unique.

8.) Find a basis for the homogenous system \((\lambda I - A)x = 0\) where \( \lambda = 2 \) and

\[
A = \begin{pmatrix}
1 & 1 & -2 \\
-1 & 2 & 1 \\
0 & 1 & -1
\end{pmatrix}.
\]

Hint: rref
\[
\begin{pmatrix}
1 & -1 & 2 \\
-1 & 1 & -1 \\
0 & -1 & 3
\end{pmatrix}.
\]

= \[
\begin{pmatrix}
1 & 0 & -1 \\
-1 & 1 & -3 \\
0 & 0 & 0
\end{pmatrix}.
\]

8.) Let \( S = \{x_1, x_2, x_3\} \) where \( x_1 = \left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right), \) \( x_2 = \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right), \) and \( x_3 = \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right). \)

a. Prove that \( S \) is an orthonormal set.

b. Express \( x = (3, 4, 5) \) in terms of the \( S \) basis.

In \( \mathbb{R}^3 \) let \( V = \{v_1, v_2, v_3\} \) where

\[ v_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \]

, \[ v_2 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \]

, and

\[ v_3 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \]

. Find the transition matrix from the \( N \)–basis to the \( V \)–basis.

In \( \mathbb{R}^3 \) let \( W = \{w_1, w_2, w_3\} \) where

\[ w_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \]
\[ w_2 = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} \]

, and

\[ v_3 = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \]

a. Write each of the \( \{v_1, v_2, v_3\} \) in terms of the:
   a. Natural basis.
   b. The \( W \) basis.
   c. The \( V \) basis.

b. Write each of the \( \{w_1, w_2, w_3\} \) in terms of the:
   a. Natural basis.
   b. The \( W \) basis.
   c. The \( V \) basis.

c. Find the transition matrix from the \( W \)–basis to the \( V \)–basis. Illustrate its effect.

d. Find the transition matrix from the \( V \)–basis to the \( W \)–basis. Illustrate its effect.

II. Diagonalization and Jordan Form

1. State and prove necessary and sufficient conditions for Diagonalization. Given a

\[
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\]

matrix find its Diagonal Form and the \( P \) which places into that form.

2.) Show that the following matrix is not diagonalizable

\[
A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}
\]

3.) define and prove properties of the minimal poly and characteristic poly.

4.) Given a 3x3 matrix find its Jordan Form and the \( P \) which places into that form.

5.) For \( W \) a subspace find \( W^\perp \) and prove that it is a subspace and that \( V \) can be written as a direct sum.

6.) Relation between \( N(A) \) and ...

III. Homework